Combinatorial Methods
in Topology and Algebra

Program of Talks

Monday

Anders Björner – **Positive sum set systems**
Let $x_1, x_2, ..., x_n$ be real numbers summing to zero, and let P be the family of all subsets $J \subseteq [n] := \{1, 2, ..., n\}$ such that $\sum_{j \in J} x_j \geq 0$. We discuss some structural and enumerative properties of such set systems. It is proved that P determines a triangulated ball whose h-polynomial is the classical Eulerian polynomial. This is used to derive lower bounds for the number of k-element subsets in P, a problem that has been studied by several authors.

Günter Ziegler – **On highly regular embeddings**

A *k-regular embedding* $X \to \mathbb{R}^N$ maps any k pairwise distinct points in a topological space X to k linearly independent vectors. The study of the existence of k-regular maps was initiated by Borsuk in 1957 and latter attracted additional attention due to its connection with approximation theory. The problem and its extensions were extensively studied by Chisholm, Cohen, Handel, and others in the 1970's and 1980's, and then again by Handel and Vassiliev in the 1990.

Our main result on k-regular maps is the following lower bound for the existence of such maps between Euclidean spaces, in which $\alpha(k)$ denotes the number of ones in the dyadic expansion of k:

Theorem. *For any $d \geq 1$ and $k \geq 1$ there is no k-regular map $\mathbb{R}^d \to \mathbb{R}^N$ for $N < d(k - \alpha(k)) + \alpha(k)$.*

This reproduces a result of Chisholm (1979) for the case of d being a power of 2; for the other values of d our bounds are in general better than Karasev's (2010), who had only recently gone beyond Chisholm's special case. In particular, our lower bound turns out to be tight for $k = 3$.

The framework of Cohen & Handel (1978) relates the existence of a k-regular map to the existence of a specific inverse of an appropriate vector bundle. Thus non-existence of regular maps into $\mathbb{R}^N$ for small N follows from the non-vanising of specific dual Stiefel–Whitney classes. This we prove using the general Borsuk–Ulam–Bourgin–Yang theorem combined with a key observation by Hung (1990) about the cohomology algebras of configuration spaces.

Our study produces similar topological lower bound results also for the existence of *ℓ-skew embeddings* $\mathbb{R}^d \to \mathbb{R}^N$ for which we require that the images of the tangent spaces of any ℓ distinct points are skew affine subspaces. This extends work by Ghomi & Tabachnikov (2008) for $\ell = 2$.

(Joint Work with Pavle V. M. Blagojević and Wolfgang Lück)

Andrea Sportiello – **Deterministic Abelian Sandpile model and square-triangle tilings**
I will recall briefly why the Abelian Sandpile Model (ASM) is, in my opinion, a very basic object in mathematics, as basic as the Poisson equation $D^2 f = g$. The ASM theory provides a remarkable theorem for the unicity of the solution, in a discretised setting where this is not evident. I will also recall why tiling models with incommensurable tiles (á-la-Penrose) are sometimes very nice, and why the one using squares and triangles is somewhat "the nicest one", with special surprising connections, e. g. with Littlewood-Richardson coefficients in the theory of symmetric functions. I will finally show that various lattice realisations of the ASM, under various circumstances, have special solutions of the associated Poisson equation, which are fractals with the topology of a Sierpinski gasket, in which each region of the gasket is filled with a different

periodic tiling. These different lattice realisations fall under one roof, once we realise that they arise through different projection procedures of a unique, "universal" solution, that lives at the level of the square-triangle tiling model.
This work is in collaboration with S. Caracciolo and G. Paoletti.

MARIO MARIETTI – **Root polytopes of finite irreducible cristallographic root systems**
Let Φ be a finite crystallographic irreducible root system and $\mathcal{P}_\Phi$ be its root polytope, that is the convex hull of the roots in Φ. We give a uniform explicit description of the polytope $\mathcal{P}_\Phi$. Through the analisys of the combinatorial structure of its faces, we can show several connections with different objects, such as, for example, the Borel subalgebras of the associated Lie algebra and certain hyperplane arrangements.

KAROLA MÉSZÁROS – **Product formulas for the volumes of flow polytopes**
I will present product formulas for the volumes of a special class of flow polytopes. This class is inspired by the Chan–Robins–Yuen polytope, whose volume is equal to a product of Catalan numbers (although there is no known combinatorial proof of this fact!). I will explain how certain algebras defined by Kirillov encode subdivisions of flow polytopes. Finally, I will show how to derive identities involving the Kostant partition function, using a connection with flow polytopes discovered by Postnikov and Stanley.

FABRIZIO CASELLI – **Isomorphisms and automorphisms of projective reflection groups**
The main motivation of this work was to investigate the generalized involution models of the projective reflection groups $G(r,p,q,n)$. This family of groups parametrizes all quotients of the complex reflection groups $G(r,p,n)$ by scalar subgroups. Our classification is ultimately incomplete, but we provide several necessary and sufficient conditions for generalized involution models to exist in various cases. In the process we have been led to consider and solve several intermediate problems concerning the structure of projective reflection groups. We derive a simple criterion for determining whether two groups $G(r,p,q,n)$ and $G(r,p',q',n)$ are isomorphic. We also describe explicitly the form of all automorphisms of $G(r,p,q,n)$, outside a finite list of exceptional cases. Building on prior work, this allows us to prove that $G(r,p,1,n)$ has a generalized involution model if and only if $G(r,p,1,n) \cong G(r,1,p,n)$. We also classify which groups $G(r,p,q,n)$ have generalized involution models when $n=2$, or q is odd, or n is odd.
This is joint work with E. Marberg (MIT).

Tuesday

Frank Sottile – **Galois groups of Schubert problems**
Work of Jordan from 1870 showed how Galois theory can be applied to enumerative geometry. Hermite earlier showed that the equivalence of Galois groups with geometric monodromy groups, and in 1979 Harris used this to study Galois groups of many enumerative problems. Vakil gave a geometric-combinatorial criterion that implies a Galois group contains the alternating group. With Brooks and Martin del Campo, we used Vakil's criterion to show that all Schubert problems involving lines have at least alternating Galois group. White and I have given a new proof of this based on 2-transitivity. My talk will describe this background and sketch a current project to systematically determine Galois groups of all Schubert problems of moderate size on all small classical flag manifolds, investigating at least several million problems. This will use supercomputers employing several overlapping methods, including combinatorial.

Alex Fink – **Matroids over rings**
Matroids are widely used objects in combinatorics; they arise naturally in many situations featuring vector configurations over a field. But in some contexts the natural data are elements in a module over some other ring, and there is more than simply a matroid to be extracted. In joint work with Luca Moci, we have defined the notion of matroid over a ring to fill this niche. I will discuss two examples of situations producing these enriched objects, one relating to subtorus arrangements producing matroids over the integers, and one related to tropical geometry producing matroids over a valuation ring. I'll also discuss the analogue of certain matroid operations and of the Tutte invariant.

Alex Engström – **Powers of monomial ideals and the asymptotic polytopes of Betti diagrams**
For any fixed monomial ideals the resolution of high enough powers are predictable. To actually gain explicit information about the stable behavior of projective resolutions of high powers is rather non-trivial if the ideals aren't particularly well behaved. Right now, discrete Morse theory for cellular resolutions seems to be the weapon of choice, and I'll first discuss recent results with Patrik Norén using that. After that I'll describe how the asymptotic decomposition of Betti tables of high powers can be conjecturally described using polytopes as a new invariant for the stable regime.

Eran Nevo – **Bipartite rigidity**
We start developing a bipartite rigidity theory for bipartite graphs, parallel to the classical rigidity theory for general graphs. Specifically, we establish bipartite analogues of the cone, contraction, deletion and gluing lemmas. We apply this theory to prove a criterion for planarity of bipartite graphs, discuss a potential application to Jockusch's cubical lower bound conjecture, and relate to Laman type results by Whiteley.
Extension to higher dimensional balanced complexes will be discussed as well, in particular a balanced analogue of the g-conjecture and Heawood inequalities.
Reflects joint work with Gil Kalai and Isabella Novik.

Raman Sanyal – **An Alexander-type duality for valuations**
Ehrhart theory is an indispensable tool in numerous areas including geometric combinatorics, commutative algebra, algebraic geometry, and approximation theory. Ehrhart and Stanley gave a generalization of the famous Ehrhart–Macdonald reciprocity to a relative setting in which certain geometrically defined subcomplexes in the boundary are related to each other. Miller and Reiner extended this to the larger class of Cohen–Macaulay complexes by casting the setup into the language of combinatorial commutative algebra. In this talk we will re-examine these results from the perspective of convex geometry and combinatorial topology. We are rewarded by extensions to more general complexes but also more general valuations, simpler proofs, and reminiscences of Alexander duality.
This is joint work with Karim Adiprasito.

Martin Tancer – **Shellability of the pinched Veronese posets**
The pinched Veronese poset $V*_n$ is the poset with ground set consisting of all non-negative integer vectors of length n such that the sum of their coordinates is divisible by n, with exception of the vector $(1,...,1)$. For two vectors a, b in $V*_n$ we have $a < b$ if and only if $b - a$ is a nonzero vector which belongs to the ground set of $V*_n$. We will discuss shellability of intervals in $V*_n$. For this we develop a new method for showing that a poset is shellable. This method differs from classical lexicographic shellability. Shellability of intervals in $V*_n$ has consequences in commutative algebra. As a corollary we obtain an alternative proof that the related pinched Veronese ring is Koszul for n at least 4 (this also follows from a result of Conca, Herzog, Trung and Valla).

Wednesday

Graham Denham – **Duality properties for abelian covers**

In parallel with a classical definition due to Bieri and Eckmann, say an FP group G is an *abelian duality group* if $H^p(G, Z[G^{ab}])$ is zero except for a single integer $p = n$, in which case the cohomology group is torsion-free. We make an analogous definition for spaces. In contrast to the classical notion, the abelian duality property imposes some direct constraints on the Betti numbers of abelian covers.
While related, the two notions are inequivalent: for example, surface groups of genus at least 2 are (Poincaré) duality groups, yet they are not abelian duality groups. Some other families of fundamental groups and spaces possess both properties: e. g. , hyperplane arrangement complements and Cohen–Macaulay toric complexes. The unifying notion is a general, cohomological vanishing result for spaces that have a Cohen–Macaulay combinatorial cover, in a sense which we make precise.
This is joint work with Alex Suciu and Sergey Yuzvinsky.

Martina Kubitzke – **Algebraic properties of path ideals of posets**

We consider path ideals associated to special classes of posets such as tree posets and cycles. We express their property of being sequentially Cohen–Macaulay in terms of the underlying poset. Moreover, monomial ideals, which arise from the Luce-decomposable model in algebraic statistics, can be viewed as path ideals of certain posets. We study invariants of these so-called Luce-decomposable monomial ideals for diamond posets and products of chains. In particular, for these classes of posets, we explicitly compute their Krull dimension, their projective dimension, their regularity and their Betti numbers.
This is joint work with Anda Olteanu.

Karim Adiprasito – **Minimal CW models for complements of 2-arrangements**

A model for a topological space is a CW complex homotopy equivalent to it. In the best case, such models are chosen to be minimal, that is, they are chosen such that the number of i-cells of the model equals the i-th rational Betti number of the space. Unfortunately, not all spaces admit minimal models. In my talk, I will investigate the question whether complements of certain subspace arrangements admit minimal models. Previous work of Hattori, Dimca–Papadima, Randell and others answered this question positively for complex hyperplane arrangements. I will demonstrate a generalization of their results to the class of 2-arrangements introduced by Goresky and MacPherson. The main idea is to establish a Lefschetz-type hyperplane theorem for complements of 2-arrangements using discrete Morse theory of Forman and the theory of combinatorial stratifications of Björner and Ziegler.

Masahiko Yoshinaga – **Milnor fibers of line arrangements**

For a given real line arrangement, we introduce a discrete-geometric objects, "the standing waves on k-resonant bands", which are elements of the vector space spanned by chambers. We can compute the monodromy eigenspaces of the Milnor fiber of the complexified line arrangement by using linear relations among standing waves. The notion standing waves directly connects the real figure with the monodromy eigenspace. We also give several conjectures on the combinatorial structures of simplicial arrangements.
This talk is based on `arXiv:1301.1430` and `arXiv:1301.1888`.

Thursday

Isabella Novik – **Balanced manifolds and pseudomanifolds**

A simplicial $(d-1)$-dimensional complex C is called *balanced* if the graph of C (i. e. , the 1-dimensional skeleton) is d-colorable. In this talk I will discuss some recent results on face numbers of balanced manifolds and pseudomanifolds, as well as present constructions of balanced manifolds (with and without boundary) with few vertices.
Parts of this work are joint with Steve Klee.

Satoshi Murai – **cd-index for CW-posets**

The flag f-vector is a basic combinatorial invariant of ranked posets, which counts the number of chains. For an Eulerian poset, its flag f-vector is efficiently encoded by a certain non-commutative polynomial, called the **cd**-index. In particular, the **cd**-indices of (the face posets of) polytopes and Gorenstein* posets are known to be non-negative and are of great interest in enumerative combinatorics.
A CW-poset is a face poset of a (finite) regular CW-complex. In this talk, I will introduce the notion of the **cd**-index to CW-posets which are not necessary Eulerian, and show that it is non-negative when a CW-poset is Cohen–Macaulay. As a corollary, I will show that the h-vector of the barycentric subdivision of a Cohen–Macaulay finite regular CW-complex is unimodal.
This is joint work with Kohji Yanagawa.

Steven Klee – **A combinatorial classification of Buchsbaum simplicial posets**

The family of Buchsbaum simplicial posets over a field $\mathbb{K}$ provides an algebraic abstraction of the family of ($\mathbb{K}$-homology) manifold triangulations. In 2008, Novik and Swartz established lower bounds on the face numbers of a Buchsbaum simplicial poset as a function of its dimension and the dimension of its $\mathbb{K}$-homology spaces; and they conjectured that these lower bounds are sufficient to classify face numbers of Buchsbaum simplicial posets with prescribed Betti numbers. We prove this conjecture by using methods from the theory of (pseudo)manifold crystallizations to construct simplicial posets with prescribed face numbers and Betti numbers.
This is joint work with Jonathan Browder.

Aldo Conca – **Universal Gröbner bases and maximal minors**

Bernstein, Sturmfels and Zelevinsky proved in 1993 that the maximal minors of a matrix of variables form a universal Gröbner basis. We present a very short proof of this result, along with a broad generalization to matrices with multi-homogeneous structures. Our main tool is a rigidity statement for radical Borel-fixed ideals in multigraded polynomial rings.

Matteo Varbaro – **Special features of subspace arrangements defined by quadratic ideals**

I will discuss the relationship between a conjecture of Eisenbud, Green and Harris on the Hilbert functions of ideals generated by quadrics and one of Kalai concerning f-vectors of Cohen-Macaulay flag complexes. In collaboration with Caviglia and Constantinescu, we obtained a solution of the EGH conjecture for monomial quadratic ideals, yielding the h-vector version of the conjecture of Kalai. One of the crucial steps in the proof seems to be related to the structure of the dual graph of a flag complex. Recently Adiprasito and Benedetti gave a sharp upper bound for the diameter of such a graph when the flag complex is normal, thereby proving the flag version of the Hirsch conjecture (which is false in general). I will end by discussing the progress done in collaboration with Benedetti on possible generalizations of the Hirsch conjecture to subspace arrangements defined by quadratic ideals.

June Huh – **Questions on positivity of algebraic cycles in toric varieties**

Rota's conjecture predicts that the coefficients of the characteristic polynomial of a matroid form a log-concave sequence. I will outline a proof for representable matroids using Milnor numbers and the Bergman fan. The same approach to the conjecture in the general case (for possibly non-representable matroids) leads to several intriguing questions on higher codimension algebraic cycles in toric varieties.

Friday

Francesco Brenti – **Dual Bayer-Billera relations, paths in the Bruhat graph, and Kazhdan–Lusztig polynomials**
We find explicitly the relations obtained by applying the inclusion-exclusion linear transformation to the Bayer–Billera (generalized Dehn–Sommerville) relations. As an application of our results we obtain a non-recursive formula for the Kazhdan–Lusztig polynomials which holds in complete generality. We conjecture that this formula cannot be simplified and give evidence in favor of this conjecture.
This is joint work with Fabrizio Caselli.

Mirkó Visontai – **Zeros of Eulerian polynomials are real**
Eulerian polynomials are descent generating polynomials that can be defined for all finite Coxeter groups. In 1994, Francesco Brenti conjectured that all zeros of these polynomials are real. In this talk, we prove this conjecture by exploiting a recurrence satisfied by refinement of these polynomials.
This is joint work with Carla D. Savage.

Sylvie Corteel – **Super-Schur functions and domino tilings**
In this work we want to enumerate infinite families of domino tilings in a strip; that we call steep tilings. To do this we use bijective combinatorics, vertex operators, Schur processes defined by Okounkov and Reshetikin and the combinatorics of Super-Schur functions due to Francesco Brenti. The key ingredient are the generalizations of the Cauchy identities for Super-Schur functions. This allows us to count the tilings, generate them randomly and compute their limit shapes. These tilings include the tilings of the Aztec diamond and the pyramid partitions, which are related to Donaldson-Thomas invariants of orbifolds thanks to some work of Ben Young.

This is joint work with Jérémie Bouttier and Guillaume Chapuy for the combinatorics and also Cédric Boutillier, Sanjay Ramassamy and Mirjana Vuletic for the asymptotics.

Mario Salvetti – **Some combinatorial constructions and relations with Artin groups**
We introduce some general combinatorial structures over posets (and, in particular, over simplicial complexes) and we find some interesting connections with (generalized) discrete Morse theory and with topological properties of Artin groups.

Petter Bränden – **Multivariate Eulerian polynomials and exclusion processes**
Recently natural multivariate extensions of Eulerian polynomials have appeared in two different settings: In the combinatorial description of the stationary distributions of the PASEP model, and in the solution of the Monotone Column Permanent Conjecture. The PASEP is a Markov process which models particles jumping on a discrete interval. Corteel and Williams proved that the stationary distribution of the PASEP (with certain parameters) may be described in terms of multivariate Eulerian polynomials (of type A). We generalize this and prove that the stationary distribution (with other parameters) may be described in terms of multivariate Eulerian polynomials of type B, and more generally in terms of multivariate Eulerian polynomials for the wreath product of the symmetric group with a cycle. We also describe strong correlation inequalities satisfied by the stationary distributions. These are obtained from precise information about the zero locus of the multivariate Eulerian polynomials in question. This generalizes and explains recent results of Hitczenko and Janson.
This is joint work with Madeleine Leander and Mirkó Visontai.

Claudio Procesi – **The energy graph of the non-linear Schrödinger equation**
Given a set S of integral points in n dimension in general position there is a graph with vertices the integral points and edges corresponding to rectangles having two vertices in the set S chosen. The set S has the meaning of *excited frequencies* and the combinatorics of the graph reflects on the nature of small solutions of the non linear equation close to the solutions with the given excited frequencies of the linear equation. A mixture of algebra, combinatorics and analysis gives a very rich picture.